Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 5.49

(a)

$F : R^3 \to R^2$ is defined by $F(x, y, z) = (x + 2y - 3z, 4x - 5y + 6z)$

(See also Problem 5.10). We will argue via matrices. Writing the vectors as columns, the mapping F can be written as $F(v) = Av$, where $v = [x, y, z]^T$ and

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$$

Using the properties of matrices:

$F(v + w) = A(v + w) = Av + Aw = F(v) + F(w)$

$F(kv) = A(kv) = kA(v) = kF(v)$

Therefore, F is linear

(b)

$F : R^2 \to R^2$ is defined by $F(x, y) = (ax + by, cx + dy)$

This problem can also be solved using matrices. However, we will solve it by directly verifying the two conditions (vector addition and scalar multiplication).

Let $v = (x_1, y_1)$ and $w = (x_2, y_2)$

$$\begin{aligned} F(v+w) &= F(x_1 + x_2, y_1 + y_2) \\ &= (a(x_1 + x_2) + b(y_1 + y_2), c(x_1 + x_2) + d(y_1 + y_2)) \\ &= ((ax_1 + by_1) + (ax_2 + by_2), (cx_1 + dy_1) + (cx_2 + dy_2)) \\ &= (ax_1 + by_1, cx_1 + dy_1) + (ax_2 + by_2, cx_2 + dy_2) \\ &= F(v) + F(w) \end{aligned}$$

$$\begin{aligned} F(kv) &= F(kx_1, ky_1) \\ &= (k(ax_1 + by_1), k(cx_1 + dy_1)) \\ &= (kax_1 + kby_1, kcx_1 + kdy_1) \\ &= k(ax_1 + by_1, cx_1 + dy_1) \\ &= kF(v) \end{aligned}$$

Therefore, F is linear

Problem 5.50

(b)

$F : R^3 \to R^2$ is defined by $F(x, y, z) = (x + 1, y + z)$

(See also Problem 5.11b)

Since $F(0, 0, 0) = (1, 0) \neq (0, 0)$, i.e. zero vector is not mapped into the zero vector, F can't be linear. (Note: Every linear mapping takes a zero vector into the zero vector, put $k = 0$ in scalar multiplication condition and verify)

(d)

$F : R^3 \to R^2$ is defined by $F(x, y, z) = (|x|, y + z)$

(See also Problem 5.11c)

Let $v = (1,0,0)$ and $k = -1$ then $F(kv) = F(-1,0,0) = (1,0)$

$kF(v) = kF(1,0,0) = -1(1,0) = (-1,0) \neq F(kv)$. So F is not linear

Problem 5.51

The mapping from $R^2 \to R^2$ is defined by a 2x2 matrix, e.g. $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$.

Therefore,

$F(1,2) = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. And,

$F(0,1) = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solving,

$x + 2y = 3$ and $0x + y = 2$, gives, $y = 2$ and $x = -1$

$z + 2t = -1$ and $0z + t = 1$, gives, $t = 1$ and $z = -3$

Hence, $\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix}$

$F(a,b) = \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = (-a+2b, -3a+b)$

Note: {(1,2),(0,1)} is a basis of R^2, so such a linear map F exists and is unique.

See also Problem 5.14.

Alternate Solution

Write (a,b) as a linear combination of $(1,2)$ and $(0,1)$ using unknowns x and y.

$(a,b) = x(1,2) + y(0,1) = (x, 2x+y)$, i.e. $a = x, b = 2x + y$

Solving for x, y in terms of a and b gives, $x = a$ and $y = -2a + b$

$$\begin{aligned} F(a,b) &= xF(1,2) + yF(0,1) \\ &= a(3,-1) + (-2a+b)(2,1) = (3a - 4a + 2b, -a - 2a + b) = (-a + 2b, -3a + b) \end{aligned}$$

Problem 5.52

(a)

The mapping $R^2 \to R^2$ is defined by a 2x2 matrix as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Solving:

$a + 3b = -2$ and $a + 4b = 3$, gives, $b = 5, a = -17$

$c + 3d = 5$ and $c + 4d = -1$, gives, $d = -6, c = 23$

$$\text{So, } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -17 & 5 \\ 23 & -6 \end{bmatrix}$$

Alternate Solution

Express a general vector (a, b) as a linear combination of the basis vectors $(1, 3), (1, 4)$ using unknowns x and y.

$(a, b) = x(1, 3) + y(1, 4) = (x + y, 3x + 4y)$. i.e. $a = x + y$ and $b = 3x + 4y$

Solving for x and y gives, $x = 4a - b$ and $y = -3a + b$

Using the linearity of the unknown T:

$$\begin{aligned}
T(a,b) &= T[(4a-b)(1,3)+(-3a+b)(1,4)] \\
&= (4a-b)T(1,3)+(-3a+b)T(1,4) \\
&= (4a-b)(-2,5)+(-3a+b)(3,-1) \\
&= (-17a+5b, 23a-6b) \\
&= \begin{bmatrix} -17 & 5 \\ 23 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
\end{aligned}$$

(b)

The mapping from $R^2 \to R^2$ is defined by a 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

By inspection, we notice that the input vectors are scalar multiples of one another (i.e. $[2,-4]^T = -2[-1,2]^T$). As the mapping is considered to be linear, the output vectors, in this case, must also be scalar multiples of one another. But, this is not the case, therefore, the mapping is not linear and can't be expressed in the form of a 2x2 matrix.

Solution to 5.53

Find a 2x2 singular matrix that maps $[1,1]^T \to [1,3]^T$.

Remember, a singular mapping F implies that there is a vector $v \neq 0$, such that $F(v) = 0$.

Based on this definition, we require a matrix that satisfies the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v \neq 0$$

This gives the two sets of equations:

$a + b = 1$, $av_1 + bv_2 = 0$

$c + d = 3$, $cv_1 + dv_2 = 0$

One possible solution can be:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

where $v = [0, 1]^T$ satisfies the singularity condition.

Problem 5.56

In this problem and the next two a set of points S is defined [these are NOT subspaces, just subsets] by giving a constraint equation on the coordinates. You are asked to describe the image $F(S)$ and pre-image $F^{-1}(S)$ under the action of a linear operator F (or G).

In problem 5.56 $F : R^2 \to R^2$. So S, $F(S)$, and $F^{-1}(S)$ are subsets of the same vector space R^2. In order to keep things straight we'll use different variables for the three spaces: x, y for the version of R^2 containing the given set S; X, Y for the image space $F(R^2)$; and u, v for the pre-image space $F^{-1}(R^2)$. Similarly, in 5.57 and 5.58 we'll use different variables for the vectors in R^3: x, y, z for the space that contains S; X, Y, Z for the space containing the image $F(S)$; and u, v, w for the space containing the pre-image $F^{-1}(S)$.

(a)

In this we have $F(x, y) = (3x+5y, 2x+3y) = (X, Y)$, where X, Y are the image variables. S is defined by: $x^2 + y^2 = 1$ in terms of (x, y), Find S in terms of (X, Y)

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Solve for x, y in terms of X, Y

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -3X + 5Y \\ 2X - 3Y \end{bmatrix}$$

i.e., $x = -3X + 5Y, y = 2X - 3Y$

Substitute these in $S : x^2 + y^2 = 1$. We get: $F(S) = (-3X + 5Y)^2 + (2X - 3Y)^2 = 1$, which, after simplification, gives the equation defining the image $F(S) : 13X^2 - 42XY + 34Y^2 = 1$

(b)

S is defined as before in terms of the x, y variables: $x^2 + y^2 = 1$, but the transformation equation is now given by $F(u, v) = (3u + 5v, 2u + 3v) = (x, y)$. Substitute the image variables into the relationship for S and re-express it in terms of the pre-image variables:

$$\begin{aligned} (3u + 5v)^2 + (2u + 3v)^2 &= 1 \\ (9u^2 + 30uv + 25v^2) + (4u^2 + 12uv + 9v^2) &= 1 \\ 13u^2 + 42uv + 34v^2 &= 1 \end{aligned}$$

Note: Ignore the confusing text answer. And there is a typo anyway.

Problem 5.57

(a)

Let $G(x, y, z) = (x + y + z, y - 2z, y - 3z) = (X, Y, Z)$, where X, Y, Z are image variables. Given $S_2 : x^2 + y^2 + z^2 = 1$ in terms of (x, y, z), find S_2 in terms of (X, Y, Z).

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solve for x, y, z in terms of X, Y, Z.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Hence, $x = X - 4Y + 3Z, y = 3Y - 2Z, z = Y - Z$.

Substitute these in $S_2 : x^2 + y^2 + z^2 = 1$ to get the constraint on X, Y, Z defining $G(S_2)$:

$$\begin{aligned} (X - 4Y + 3Z)^2 + (3Y - 2Z)^2 + (Y - Z)^2 &= 1 \\ X^2 - 8XY + 26Y^2 + 6XZ - 38YZ + 14Z^2 &= 1 \end{aligned}$$

(b)

S_2 is defined as before in terms of the x, y, z variables: $x^2 + y^2 + z^2 = 1$, but the transformation equation is now given by $G(u, v, w) = (u+v+w, v-2w, v-3w) =$

(x, y, z). Substitute the image variables into the relationship for S_2 and re-express it in terms of the pre-image variables:

$$\begin{aligned}
1 &= x^2 + y^2 + z^2 \\
1 &= (u + v + w)^2 + (v - 2w)^2 + (v - 3w)^2 \\
1 &= (u^2 + v^2 + w^2 + 2uv + 2uw + 2vw) + (v^2 - 4vw + 4w^2) + (v^2 - 6vw + 9w^2) \\
1 &= u^2 + 3v^2 + 14w^2 + 2uv + 2uw - 8vw
\end{aligned}$$

Problem 5.58

(a)

Same notation and linear operator G as in 5.57. The set of points is a plane (but it's **NOT** a subspace because $(0,0,0) \notin H$) $H : x + 2y - 3z = 4$ in terms of x, y, z. For this we have to find H in terms of the image X, Y, Z variables.

From 5.57(a), we have $x = X - 4Y + 3Z, y = 3Y - 2Z, z = Y - Z$, so substitute these in the H equation to get the equation defining $G(H) : (X - 4Y + 3Z) + 2(3Y - 2Z) - 3(Y - Z) = 4$, which simplifies to $X - Y + 2Z = 4$. Note that the image of H is also a plane, but it is not a subspace.

(b)

Same as in 5.57, but use the H set. Substitute the image variables into the relationship for H and re-express it in terms of the pre-image variables (u, v, w):

$$\begin{aligned}
4 &= (u + v + w) + 2(v - 2w) - 3(v - 3w) \\
4 &= u + v(1 + 2 - 3) + w(1 - 4 + 9) \\
4 &= u + 6w
\end{aligned}$$

The pre-image is also a plane, but not a subspace, Note: There is a typo in the text answer for this question.

Problem 5.70

$F(x, y, z) = (y, x + z)$ and $G(x, y, z) = (2z, x + y)$

- $F + G$

$$\begin{aligned}(F+G)(x,y,z) &= F(x,y,z) + G(x,y,z) \\ &= (y, x+z) + (2z, x+y) = (y+2z, 2x+y+z)\end{aligned}$$

- $3F - 2G$

$$\begin{aligned}(3F-2G)(x,y,z) &= 3F(x,y,z) - 2G(x,y,z) \\ &= 3(y, x+z) - 2(2z, x+y) = (3y-4z, x-2y+3z)\end{aligned}$$

Note: There is a typo in the text answer for this question.

Problem 5.71

$F(x, y, z) = (y, x + z)$, $G(x, y, z) = (2z, x + y)$, and $H(x, y) = (y, 2x)$

(a) $H \circ F$ and $H \circ G$

$$\begin{aligned}H \circ F(x,y,z) &= H(F(x,y,z)) \\ &= H(y, x+z) \\ &= (x+z, 2y)\end{aligned}$$

$$\begin{aligned}H \circ G(x,y,z) &= H(G(x,y,z)) \\ &= H(2z, x+y) \\ &= (x+y, 4z)\end{aligned}$$

(b) $F \circ H$ and $G \circ H$

The mappings $F \circ H$ and $G \circ H$ are not defined since the image of H is not contained in the domain of F and G.

(c) $H \circ (F + G)$ and $H \circ F + H \circ G$

$$\begin{aligned}(H \circ (F+G))(x,y,z) &= H((F+G)(x,y,z)) \\ &= H(F(x,y,z)+G(x,y,z)) \\ &= H((y, x+z)+(2z, x+y)) \\ &= H(y+2z, 2x+y+z) \\ &= (2x+y+z, 2y+4z)\end{aligned}$$

$$\begin{aligned}(H \circ F + H \circ G)(x,y,z) &= (H \circ F)(x,y,z) + (H \circ G)(x,y,z) \\ &= (x+z, 2y) + (x+y, 4z) \\ &= (2x+y+z, 2y+4z)\end{aligned}$$

Problem 5.61

(a)

The image (or range) of $F : R^3 \to R^3$ is the same as the column space of the defining matrix A, if we consider the map as a matrix transformation F_A.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 1 & 4 & 1 \end{bmatrix}$$

To obtain a basis for the image, row-reduce A to echelon form and use the 'casting-out' algorithm:

$$A \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 of A are a basis for the column space [the corresponding columns in the echelon form of A having pivots]. Then $\dim(\text{Im } F) = 2$ with basis $\{(1,2,1),(2,5,4)\}$.

Note: The answer in the book is obtained by directly reducing the column vectors of A as vector in R^3, and it's equivalent since $(2,5,4) = 2(1,2,1) + (0,1,2)$

$\dim(\text{Ker } F) = \dim(R^3)$- $\dim(\text{Im } F) = 3 - 2 = 1$.

To find a basis for Ker F solve $F(x,y,z) = (0,0,0)$:

$$\begin{aligned} x + 2y - 3z &= 0 \\ 2x + 5y - 4z & \quad 0 \\ x + 4y + z &= 0 \end{aligned}$$

The coefficient matrix has already been reduced in part a) above [that's the advantage of doing the casting out method to get the image basis]:

$$A \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the free variable is z, dim(Ker F)=1. The general solution for $F(x,y,z) = 0$ is given by $(7z, -2z, z) = z(1,-2,1)$ so a basis for Ker(F) is $\{(7,-2,1)\}$

Note: these answers for bases of the image and kernel are not unique of course.

(b)

First find the image of the usual basis vectors of R^4

$$\begin{aligned}
F(1,0,0,0) &= (1,2,1)\\
F(0,1,0,0) &= (2,4,2)\\
F(0,0,1,0) &= (3,7,6)\\
F(0,0,0,1) &= (2,5,5)
\end{aligned}$$

The image vectors span ImF. Form a matrix A whose rows are these image vectors and row reduce to echelon form:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 6 \\ 2 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$(1,2,1)$ and $(0,1,3)$ form a basis of ImF. Hence dim(ImF)=2

For the kernel we have dim(Ker F)= dim(R^4)- dim(ImF) $= 4-2=2$.
For a basis of ker$F$ we need to solve

$$\begin{aligned}
x+2y+3z+2t &= 0\\
2x+4y+7z+5t &= 0\\
x+2y+6z+5t &= 0
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 7 & 5 \\ 1 & 2 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
x+2y+3z+2t &= 0\\
z+t &= 0
\end{aligned}$$

The free variables are y and t. The solution is:

$(x, y, z, t) = (t - 2y, y, -t, t) = t(1, 0, -1, 1) + y(-2, 1, 0, 0)$

So for the basis of Ker$F$ we can choose: $\{(1, 0, -1, 1), (-2, 1, 0, 0)\}$

Note: these answers for bases of the image and kernel are not unique or course.

Problem 5.62

(b)

Find the images of usual basis vectors of R^3:

$$\begin{aligned} G(1,0,0) &= (1,0) \\ G(0,1,0) &= (1,1) \\ G(0,0,1) &= (0,1) \end{aligned}$$

These image vectors span ImG. Form a matrix A whose rows are these image vectors, and row reduce to echelon form:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$(1, 0), (0, 1)$ form the basis of ImG, dim(ImG)=2, i.e. ImG $= R^2$.

For the kernel we have dim(Ker$G$)= dim(R^3)- dim(ImG) $= 3 - 2 = 1$.

For a basis of Ker$G$ we need to solve

$$\begin{aligned} x + y &= 0 \\ y + z &= 0 \end{aligned}$$

The solution is:

$x = -y$, $z = -y$, or $(x, y, z) = (-y, y, -y) = y(-1, 1, -1)$.

For a basis for the Ker$G$ we can choose: $\{(-1,1,-1)\}$

Note: These answers for bases of the image and kernel are not unique of course.

(c)

The image (ImG) is the same as the column space of the defining matrix A, if we consider the map as a matrix transformation G_A.

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 2 & -1 \\ 3 & 6 & 8 & 5 & -1 \end{bmatrix}$$

To obtain a basis for the image, i.e. the column space of A, we row-reduce A and use the casting-out algorithm with columns:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & 2 & -1 \\ 3 & 6 & 8 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M$$

Therefore, $\dim(\mathrm{Im}G) = 2$, and a basis for the image ImG consists of columns 1 and 3 of A, i.e. $\{(1,1,3),(2,3,8)\}$.

For the kernel we have dim(Ker G)= $\dim(R^5)$-dim(ImG) $= 5 - 2 = 3$.

For a basis of Ker$G$ we need to solve

$$\begin{aligned} x + 2y + 2z + s + t &= 0 \\ x + 2y + 3z + 2s - t &= 0 \\ 3x + 6y + 8z + 5s - t &= 0 \end{aligned}$$

But we already row-reduced A to get the image basis, so we don't need to repeat that [that's the advantage of using the column casting-out method to get the image, instead of working with the transpose and rows - less work if you also want a kernel basis]

Referring to the row-reduced matrix M, the free variables are y,s, and t. The solution is:

$$\begin{aligned}(x, y, z, s, t) &= (-2y + s - 5t, y, -s + 2t, s, t) \\ &= y(-2, 1, 0, 0, 0) + s(1, 0, -1, 1, 0) + t(-5, 0, 2, 0, 1)\end{aligned}$$

So for the basis of Ker$G$ we can choose:
$\{(-2, 1, 0, 0, 0), (1, 0, -1, 1, 0), (-5, 0, 2, 0, 1)\}$
Note: these answers for bases of the image and kernel are not unique of course.

Problem 5.63

(a)

The image $F(R^4)$ (image of F) is the same as the column space of the defining matrix A, if we consider the map as a matrix transformation F_A.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix}$$

To obtain a basis for the image, i.e. the column space of A, row-reduce A and use the casting-out columns method.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 2 & -1 \\ 1 & -3 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 5 & -2 & 3 \\ 0 & 5 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M$$

Therefore, $\dim(F(R^4)) = 2$, and a basis for the image $F(R^4)$ consists of columns 1 and 2 of A, i.e. $\{(1, 2, 1), (2, -1, -3)\}$.
For the kernel we have $\dim(\ker F) = \dim(R^4) - \dim F(R^4) = 4 - 2 = 2$.

For a basis of $\ker F$ we need to solve

$$\begin{aligned} x+2y+0z+t &= 0 \\ 2x-y+2z-t &= 0 \\ x-3y+2z-2t &= 0 \end{aligned}$$

The previous row-reduction of A can be applied here as well. Referring to the reduced matrix M, the free variables are z and t. The solution is:
$(x, y, z, t) = (t/5 - 4z/5, -3t/5 + 2z/5, z, t) = z/5(-4, 2, 5, 0) + t/5(1, -3, 0, 5)$.
So for the basis of $\ker F$ we can choose: $\{(-4, 2, 5, 0), (1, -3, 0, 5)\}$
Note: As always there are many possible solutions for the basis of the image and kernel. It's a good idea to get rid of fractions by subsuming them into the arbitrary constants as in the above.

(b)

The image $F(R^4)$ (image of F) is the same as the column space of the defining matrix A, if we consider the map as a matrix transformation F_A.

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 0 & -5 & 3 \end{bmatrix}$$

To obtain a basis for the image, i.e. the column space of A, row-reduce A and use the casting-out columns method.

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 0 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} = M$$

Therefore, $\dim(F(R^4)) = 3$, and a basis for the image $F(R^4)$ consists of columns 1, 2, and 3 of A, i.e. $\{(1, 2, -2), (0, 3, 0), (2, -1, -5)\}$.

For the kernel we have $\dim(\ker F) = \dim(R^4) - \dim F(R^4) = 4 - 3 = 1$.

For a basis of $\ker F$ we need to solve

$$\begin{aligned} x + 0y + 2z - t &= 0 \\ 2x + 3y - z + t &= 0 \\ -2x + 0y - 5z + 3t &= 0 \end{aligned}$$

The previous row-reduction of A can be applied here as well. Referring to the reduced matrix M, the one free variable is t. The solution is:

$(x, y, z, t) = (-t, 2t/3, t, t) = t/3(-3, 2, 3, 3)$

So for the basis of $\ker F$ we can choose: $\{(-3, 2, 3, 3)\}$

Note: there are many possible solutions for the basis of the image and kernel.

Note: In problems of these kind you have a choice when finding the basis for the image - either row-reduce A and cast out columns of A that don't have pivots in the reduced matrix, OR row-reduce A^T and use the non-zero rows for the basis. If you also have to find a basis for the kernel it is quicker to use A and the column method, since you will have to row-reduce A anyway to solve the equations needed to find the kernel. Easier work to row-reduce just A, rather than both A and A^T.

Problem 5.64

The image (range) of a linear map is just the column space of a matrix which defines it. So write the given basis vectors as columns of a 3×3 matrix:

$$\begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

The last column can be any linear combination of the other two, so the column space has the first two columns as a basis. We have chosen the zero combination for simplicity. Any other l.c. will do. With our choice, the transformation is $F(x, y, z) = (x + 4y, 2x + 5y, 3x + 6y)$

Problem 5.65

The linear mapping, $G : R^4 \rightarrow R^3$ can be written as a 3x4 matrix A. The nullspace of A has dimension 2 [a basis with two vectors is given in the problem] and $\dim R^4 = 4$, therefore rank $A = 4 - 2 = 2$. Without loss of generality we can assume that A is in echelon form in which there are two non-zero rows:

$$A = \begin{bmatrix} a & b & c & d \\ & \cdots & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Any linear combination of the first two rows could be used for row three, but the simplest to use is the zero combination as given above. Since the kernel is the nullspace of A, the following relationships hold, using the given vectors:

$$\begin{bmatrix} a & b & c & d \\ & \cdots & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ & \cdots & & \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Looking at just the first entry in the zero image vector we can work with just the first row of A:

$$\begin{aligned} a + 2b + 3c + 4d &= 0 \\ b + c + d &= 0 \end{aligned}$$

The free variables are c and d. The solution is:
$(a, b, c, d) = (-c - 2d, -c - d, c, d) = c(-1, -1, 1, 0) + d(-2, -1, 0, 1)$.

If we repeat for the second row we'll get exactly the same result. So picking two linearly independent vectors for rows 1 and 2 from this solution we get say: $(-1,-1,1,0)$ and $(-2,-1,0,1)$. As noted above, the third row of A can be any linear combination of the first two, but we might as well choose the zero combination for simplicity. [If we chose a third linearly independent row for A we would change the rank of A to 3, in which case the nullity [dimension of the nullspace] drops to 1. But this would not be consistent with the requirement that the two given linearly independent vectors span the nullspace.]. So A can be chosen to be:

$$A = \begin{bmatrix} -1 & -1 & 1 & 0 \\ -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The resulting transformation is given by:

$F(x,y,z,t) = (-x-y+z, -2x-y+t, 0)$.

The answer for this problem is obviously not unique.

Problem 5.66

$P_{10} = a_{10}t^{10} + a_9t^9 + a_8t^8 + ... + a_1t^1 + a_0t^0$

$P_{10}^{(4)} = 10(9)(8)(7)a_{10}t^6 + 9(8)(7)(6)a_9t^9 + ... + 4(3)(2)(1)a_4t^0$

(a)

From this, the standard basis for the image is $\{1, t, t^2, ..., t^6\}$ and its dimension is therefore 7.

(b)

It also follows that the standard basis for the kernel is $\{1, t, t^2, t^3\}$ from which the dimension of the kernel is 4.